

# Bethe ansatz and Hirota equation in integrable models

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## Abstract

In this short review the role of the Hirota equation and the tau-function in the theory of classical and quantum integrable systems is outlined.

**1. Introduction.** As is known, integrable models exist in two versions – classical and quantum. In the former one should solve equations of motion while in the latter the primary concern is to diagonalize operators (Hamiltonians or evolution operators). Usually classical integrable models admit quantization that preserves integrability and quantum models have a well-defined classical limit, in accordance with the quantum-mechanical correspondence principle.

At the same time it turns out that there are deeper links between quantum and classical integrable models which do not follow from the usual correspondence principle. Namely, it appears that classical integrable equations are built in the structure of quantum models as exact relations even at  $\hbar \neq 0$ . And vice versa, some specific accessories of the quantum theory appear in solving purely classical integrable problems. There are several aspects of this surprising and not yet fully understood phenomenon. We discuss only one – the identification of the quantum transfer matrix (the  $T$ -operator) with the classical  $\tau$ -function.

In the main text we do not give any references. Some references with brief comments are collected in section 7.

**2. Quantum integrable systems: Bethe ansatz.** It is noteworthy that the first non-trivial problem with many degrees of freedom solved exactly was quantum rather than classical. In 1931 H.Bethe managed to find exact wave functions of the Heisenberg spin chain with the Hamiltonian

$$H = J \sum_{n=1}^L \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z \right).$$

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Here  $\sigma^x, \sigma^y, \sigma^z$  are the standard Pauli matrices. The key point of the solution was a special ansatz for the wave function which is now called the Bethe ansatz. Later one or other version of this method turned out to be applicable to many other lattice or continuous quantum models in  $1 + 1$  dimensions. The typical form of the answer for the spectrum of the Hamiltonian is

$$E = \sum_k \varepsilon(v_k).$$

Here  $\varepsilon(v)$  is a known function (for the Heisenberg chain it is  $\varepsilon(v) = -8J/(v^2+1)$ ) and the numbers  $v_k$  which are rapidities of quasiparticles are found from the system of algebraic equations (the Bethe equations)

$$\left( \frac{v_k - i}{v_k + i} \right)^L = - \prod_l \frac{v_k - v_l - 2i}{v_k - v_l + 2i}. \quad (1)$$

In physically interesting cases, when the ground state in the thermodynamic limit is built by filling a false vacuum and the number of quasiparticles tends to infinity together with the lattice length  $L$ , the system of Bethe equations becomes an integral equation for the density of continuously distributed rapidities which can be solved explicitly.

In the case of generalized spin chains with “spin variables” belonging to representations of the group  $SU(N)$ , diagonalization of Hamiltonian is performed by means of consecutive application of the Bethe ansatz  $N - 1$  times (this method is usually referred to as nested Bethe ansatz). Correspondingly, there are several sorts of quasiparticles with rapidities  $v_k^{(t)}$ ,  $t = 1, 2, \dots, N - 1$ , and the Bethe equations read

$$\prod_l \frac{v_k^{(t)} - v_l^{(t-1)} + i}{v_k^{(t)} - v_l^{(t-1)} - i} \frac{v_k^{(t)} - v_l^{(t)} - 2i}{v_k^{(t)} - v_l^{(t)} + 2i} \frac{v_k^{(t)} - v_l^{(t+1)} - i}{v_k^{(t)} - v_l^{(t+1)} + i} = -1. \quad (2)$$

For many years a number of different versions and generalizations of the Bethe method were suggested. However, its main secret seems to be still unrevealed.

**3. Integrable models of classical theory: Hirota equation.** It is a question about non-linear partial differential equations. Today we know many cases when they appear to be integrable. Among them are well-known examples of integrable models of classical field theory in  $1 + 1$  dimensions: sin-Gordon, the principal chiral field, sigma-models. There are also non-relativistic models such as the Korteweg - de Vries equation, the non-linear Schrodinger equation, the Kadomtsev-Petviashvili equation, the Toda chain and many other. They are often called soliton equations because they usually have exact solutions of that type (localized moving excitations that preserve their shape in the evolution). The integrability, i.e. the existence of infinite number of conserved quantities in involution, means that with each such equation one can associate an infinite *hierarchy* of compatible equations since each integral of motion generates its own evolution in time.

The central object of the theory is a function on an infinite dimensional parameter space which is called  $\tau$ -function. In a nutshell, it provides a common solution to the whole hierarchy of integrable equations generated by the infinite family of Hamiltonians in involution. As such, it depends on infinite number of variables  $\mathbf{t} = \{t_0, t_1, t_2, t_3, \dots\}$  and satisfies infinite number of equations which can be encoded in just one functional

relation (the Hirota equation)

$$\begin{aligned}
& (z_2 - z_3) \tau(t + [z_1]) \tau(t + [z_2] + [z_3]) \\
& + (z_3 - z_1) \tau(t + [z_2]) \tau(t + [z_1] + [z_3]) \\
& + (z_1 - z_2) \tau(t + [z_3]) \tau(t + [z_1] + [z_2]) = 0.
\end{aligned} \tag{3}$$

where  $z_1, z_2, z_3$  are arbitrary parameters, and we use the short-hand notation  $t+[z] \equiv \{t_0+1, t_1+z, t_2+\frac{1}{2}z^2, t_3+\frac{1}{3}z^3, \dots\}$ . Expansion in powers of  $z_i$  yields differential equations which constitute the hierarchy. (To be precise, (3) corresponds to the modified Kadomtsev-Petviashvili hierarchy; in other cases the functional relations have a similar bilinear form.) Along with  $t_i$  one may also use the variables  $u_z$  which are numbered by continuous “label”  $z$  and which are connected with the  $t_i$ ’s by the relations

$$t_0 = u_0, \quad t_k = \frac{1}{k} \sum_z u_z z^k, \quad k \geq 1 \tag{4}$$

(let us assume that the sum is finite). In the variables  $u_z$  equation (3) becomes a difference equation on a 3D lattice for any triple  $u_{z_i} = u_i$  ( $i = 1, 2, 3$ ):

$$\begin{aligned}
& (z_2 - z_3) \tau(u_1, u_2 + 1, u_3 + 1) \tau(u_1 + 1, u_2, u_3) \\
& + (z_3 - z_1) \tau(u_1 + 1, u_2, u_3 + 1) \tau(u_1, u_2 + 1, u_3) \\
& + (z_1 - z_2) \tau(u_1 + 1, u_2 + 1, u_3) \tau(u_1, u_2, u_3 + 1) = 0.
\end{aligned} \tag{5}$$

The parameters  $z_i$  play the role of the lattice spacings. Similar equations can be written for four and more variables  $u_i$  but all of them appear to be algebraic consequences of equations (5) written for any triple of variables.

The sets of variables  $t_i$  and  $u_z$  provide complimentary descriptions of one and the same integrable system and transition from one to another is in some sense similar to the Fourier transform. Let us also note that the Hirota equation written in the forms (3) or (5) reflects a deep interrelation between continuous and discrete or difference soliton equations: they belong to one and the same hierarchy and turn one into another at the change of the independent variables (4) made simultaneously in the whole hierarchy.

“In nature”  $\tau$ -functions or their logarithms appear as partition functions, different kinds of correlators and their generating functions, and effective actions as functions of coupling constants.

In fact the whole variety of integrable non-linear partial differential equations can be encoded in *one* universal difference equation (5) for the  $\tau$ -function. As well as the equivalent equation (3), it is called the Hirota equation. It plays a truly vital role in the theory of classical (and also quantum, as we shall see soon) integrable systems. All known integrable equations can be obtained from it by various simple but maybe technically sophisticated manipulations such as continuous limit (expansion in powers of  $z_i$  and transition to the variables  $t_k$ ), imposing of reductions, choosing dependent and independent variables and the like.

As any fundamental thing, the Hirota equation and closely related equation called the  $Y$ -system often appears in different unexpected contexts. Recently it was used for finding

the spectrum of anomalous dimensions of composite operators in  $N = 4$  supersymmetric 4D Yang-Mills theory in the planar limit.

**4. Towards a synthesis of classical and quantum integrability.** By quantum integrable system we mean, in this section, a non-homogeneous spin chain whose local observables (“spins” on the sites) are operators acting in finite dimensional representations of the  $su(N)$  algebra or its  $q$ -deformation. This example is rather representative because many other exactly solvable quantum models can be treated, at least formally, as its limiting cases.

Since the system has many or even infinitely many commuting integrals of motion  $H_{\{J\}}$ , where  $\{J\} = \{J_1, J_2, \dots\}$  is some multi-index, it is natural to diagonalize them simultaneously rather than one particular Hamiltonian from this family. Even better, one can combine them in a generating function and diagonalize this operator function. Schematically, it looks like

$$T(t_0, t_1, t_2, \dots) = \sum_{\{J\}} a_{\{J\}} \left( \prod_j t_j^{J_j} \right) H_{\{J\}}, \quad (6)$$

where  $a_{\{J\}}$  are properly chosen coefficients. It depends on an infinite number of auxiliary variables  $t_i$ . Such generating function is called the master  $T$ -operator (or simply  $T$ -operator). It has the meaning of an evolution operator in a time determined by the parameters  $t_i$ . The  $T$ -operator is a much more meaningful and informative object than any particular Hamiltonian because an important dynamical information is encoded in its analytic properties in the variables  $t_i$  (which are in general complex numbers).

The commuting family of operators constructed in such a way contains, along with the full set of commuting Hamiltonians of the spin chain, the transfer matrices of associated 2D lattice models of statistical mechanics (vertex models) as well as all Baxter’s  $Q$ -operators.

More often than not an explicit expression for the  $T$ -operator through original dynamical variables is not available but this is not an obstacle for the derivation of general functional relations for it. With taking into account the analytic properties, they allow one to solve the spectral problem for the  $T$ -operator and thus for the original Hamiltonian. These functional relations have a long history and are known in different forms. In a sense, they are at the top of the theory of quantum integrable systems. Remarkably, at this place the quantum theory becomes very close to the classical one.

The key fact that provides a hint for the anticipated synthesis of classical and quantum integrability is that the most general and universal form of the functional relations for the  $T$ -operator  $T(t_0, t_1, \dots)$  (6) is nothing else than the classical Hirota equation (3) in the variables  $t_i$ . Let us remark that since the  $T$ -operators commute for all values of  $t_i$ , the ordering ambiguity, usually accompanying quantization, does not arise here, and any eigenvalue of the  $T$ -operator satisfies the Hirota equation. More precisely, the  $T$ -operator satisfies the general bilinear relation for the  $\tau$ -function of the modified Kadomtsev-Petviashvili (mKP) hierarchy from which the equations of the Hirota type follow. In other words, the  $T$ -operator (or rather any of its eigenvalues) and the classical  $\tau$ -function can be identified:

$$T(t_0, t_1, t_2, \dots) = \tau(t_0, t_1, t_2, \dots).$$

To make the statement more precise, we need some details about the  $T$ -operator. Among the variables  $t_i$ , the first one,  $t_0$ , is distinguished. In order to stress this, we denote it by  $u$ :  $u = t_0$ . It is called the quantum spectral parameter. Let us consider the  $T$ -operator as a function of  $u$ , depending on all other  $\mathbf{t} = \{t_1, t_2, \dots\}$  as on parameters:  $T = T(u, \mathbf{t})$  (for definiteness, one can keep in mind any of its eigenvalues). Let  $T_\lambda(u)$  be the set of commuting quantum transfer matrices. They are functions of  $u$  and depend on a Young diagram  $\lambda$  as on discrete (multi-component) parameter. The  $T$ -operator can be represented as the following generating series which is the precise version of the schematic formula (6):

$$T(u, \mathbf{t}) = \sum_{\lambda} s_{\lambda}(\mathbf{t}) T_{\lambda}(u). \quad (7)$$

The sum is taken over all Young diagrams including the empty one and  $s_{\lambda}(\mathbf{t})$  are the Schur functions (which are usually regarded as symmetric functions of the variables  $x_i$  such that  $t_k = \frac{1}{k} \sum_i x_i^k$ ). The known functional relations for  $T_{\lambda}(u)$  imply that this object satisfies the classical mKP hierarchy and, in particular, the Hirota equation. The Planck constant  $\hbar$  of the quantum problem is not present in the Hirota equation but appears in its solutions.

In the context of the classical Hirota equation, the nested Bethe ansatz method translates to a chain of Bäcklund transformations which sequentially “undress” the original solution to a trivial one.

All this can be almost literally extended to quantum integrable models with supersymmetry. The Hirota equation for the  $T$ -operator remains the same, the only change is the class of relevant solutions.

**5. Connection with many-body problems of the Calogero-Moser type.** Another side of the classical-quantum correspondence is the link to classical integrable many-body problems of the Calogero-Moser type which enter into the game through dynamics of zeros of the  $T$ -operator. This link follows from the identification of the  $T$ -operator with the classical  $\tau$ -function and from the analytic properties of the former.

To give a more precise statement, we need to know the analytic properties of the  $T$ -operator  $T(u, \mathbf{t})$ ,  $u = t_0$ , in the variable  $u$  (the quantum spectral parameter). This is one of the most important characteristics of the quantum model. In the simplest case, which includes finite spin chains, the  $T$ -operator should be a polynomial of  $u$  of degree equal to the length of the chain:

$$T(u, \mathbf{t}) = C \prod_{j=1}^L (u - u_j(\mathbf{t})). \quad (8)$$

Then from the identification of  $T(u, \mathbf{t})$  with the  $\tau$ -function it follows that the dynamics of the zeros  $u_j(\mathbf{t})$  in the times  $t_i$  is given by the relativistic generalization of the Calogero-Moser model (it is usually called the Ruijsenaars-Schneider system). For example the equations of motion in  $t_1$  are

$$\ddot{u}_i = \sum_{k \neq i} \frac{2\dot{u}_i \dot{u}_k}{(u_i - u_k)^2 - 1}, \quad \dot{u}_i = \partial_{t_1} u_i. \quad (9)$$

This system is integrable and has the required number of independent conserved quantities  $\mathcal{H}_j$  in involution.

Finally, let us stress the specific way of posing the classical mechanical problem for the Ruijsenaars-Schneider system that corresponds to diagonalization of quantum spin chain Hamiltonians or transfer matrices of the vertex model. The standard mechanical problem is: given initial coordinates and velocities of the particles  $u_j(0)$ ,  $\dot{u}_j(0)$ , find  $u_j(t)$ . In distinction to this, in order to find eigenvalues of the  $T$ -operator one should pose the problem in the following non-standard way: given initial coordinates  $u_j = u_j(0)$  and values of all higher integrals of motion  $\mathcal{H}_j$ , find initial velocities  $\dot{u}_j(0)$ . The solution of such a problem in general is not unique: different possible solutions correspond to different eigenstates of the quantum Hamiltonians.

We also note that the Bethe equations (2) can be understood as the Ruijsenaars-Schneider system in the discrete time  $t$ .

**6. Concluding remarks.** We see that the most universal relations for classical and quantum integrable systems actually coincide and are given by the Hirota equation. Summing up, we suggest the following extension of the quantum-mechanical correspondence principle: with any quantum integrable system one can associate a classical integrable dynamics in the space of its (commuting) integrals of motion. This classical dynamics contains complete information about spectral properties of the quantum system.

At this point we finish the story about interrelations between classical and quantum integrability told “at verbal level”. In our opinion, the very fact that the top points of the classical and quantum theories of integrable systems actually coincide takes place for profound reasons and requires a deeper understanding.

**7. Some references and comments.** The fundamental role of the  $\tau$ -function in the theory of soliton equations was revealed in the works of the Kyoto school (see e.g. [1] and references therein). The discrete Hirota equation first appeared in [2], its meaning for integrable hierarchies was clarified in [3]. The Bethe ansatz is reviewed in [4, 5]. The functional relations for quantum transfer matrices in the form of determinant formulas were found in [6, 7], for the supersymmetric extension see [8]. A similarity between quantum transfer matrices and classical  $\tau$ -functions was first pointed out in [9] (see also [10]), where the discrete classical Hirota dynamics in the space of commuting quantum integrals of motion was introduced. An extension of this approach to supersymmetric models was suggested in [11]. For the master  $T$ -operator see [12, 13]. The dynamics of poles of solutions to classical integrable equations (zeros of the  $\tau$ -function) in connection with solvable many-body problems was studied in [15, 16, 17]. The coincidence of the equations of motion for the discrete time analog of the Ruijsenaars-Schneider system and the Bethe equations was noticed in [18], a connection with motion of zeros of  $\tau$ -function was pointed out in [9]. For various aspects and applications of the Hirota equation (the  $T$ -system) and the associated  $Y$ -system see review [19] and references therein. Another context where  $\tau$ -functions of classical integrable hierarchies enter the theory of quantum integrable models and associated 2D lattice models of statistical mechanics is calculation of scalar products and partition functions with domain wall boundary conditions [20].

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## References

- [1] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations*, in "Nonlinear integrable systems – classical and quantum", eds. M. Jimbo and T. Miwa, World Scientific, pp. 39-120 (1983).
- [2] R. Hirota, *Discrete analogue of a generalized Toda equation*, J. Phys. Soc. Japan **50** (1981) 3785-3791.
- [3] T. Miwa, *On Hirota's difference equations*, Proc. Japan Acad. **58** (1982) 9-12.
- [4] V. Korepin, N. Bogoliubov, and A. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, 1993.
- [5] L. D. Faddeev, *How Algebraic Bethe Ansatz works for integrable model*, arXiv:hep-th/9605187;  
L. D. Faddeev, *Algebraic aspects of Bethe Ansatz*, Int. J. Mod. Phys. A **10** (1995) 18451878, arXiv:hep-th/9404013.
- [6] I. Cherednik, *An analogue of character formula for Hecke algebras*, Funct. Anal. and Appl. **21:2** (1987) 94-95 (translation: pgs 172-174).
- [7] V. Bazhanov and N. Reshetikhin, *Restricted solid-on-solid models connected with simply laced algebras and conformal field theory*, J. Phys. A: Math. Gen. **23** (1990) 1477-1492.
- [8] Z. Tsuboi, *Analytic Bethe ansatz and functional equations for Lie superalgebra  $sl(r+1|s+1)$*  J. Phys. A **30** (1997) 7975-7991.
- [9] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, *Quantum integrable systems and elliptic solutions of classical discrete nonlinear equations*, Commun. Math. Phys. **188** (1997) 267-304.
- [10] A. Zabrodin, *Discrete Hirota's equation in quantum integrable models*, Int. J. Mod. Phys. **B11** (1997) 3125-3158;  
A. Zabrodin, *Hirota equation and Bethe ansatz*, Teor. Mat. Fyz., **116** (1998) 54-100 (English translation: Theor. Math. Phys. **116** (1998) 782-819).
- [11] V. Kazakov, A. Sorin and A. Zabrodin, *Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics*, Nucl. Phys. **B790** (2008) 345-413;  
A. Zabrodin, *Bäcklund transformations for difference Hirota equation and supersymmetric Bethe ansatz*, Teor. Mat. Fyz. **155** (2008) 74-93 (English translation: Theor. Math. Phys. **155** (2008) 567-584).
- [12] V. Kazakov, S. Leurent and Z. Tsuboi, *Baxter's Q-operators and operatorial Bäcklund flow for quantum (super)-spin chains*, Commun. Math. Phys. **311** (2012) 787-814.

- [13] A. Alexandrov, V. Kazakov, S. Leurent, Z. Tsuboi and A. Zabrodin, *Classical tau-function for quantum spin chains*, arXiv:1112.3310.
- [14] A. Zabrodin, *The master T-operator for vertex models with trigonometric R-matrices as classical tau-function*, arXiv:1205.4152.
- [15] H. Airault, H. McKean and J. Moser, *Rational and elliptic solutions of the KdV equation and related many-body problems*, Comm. Pure Appl. Math. **30** (1977) 95-125.
- [16] I. Krichever, *Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles*, Funk. Anal. i ego Pril. **14:4** (1980) 45-54 (translation: Funct. Anal. Appl., **14:4** (1980) 282-290);  
 I. Krichever, *On rational solutions of the Kadomtsev-Petviashvili equation and on integrable systems of N particles on the line*, Funct. Anal. Appl. **12:1** (1978) 76-78;  
 I. Krichever and A. Zabrodin, *Spin generalization of the Ruijsenaars-Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra*, Uspekhi Mat. Nauk, **50:6** (1995) 3-56 (translation: Russ. Math. Surv., **50:6** (1995) 1101-1150).
- [17] T. Shiota, *Calogero-Moser hierarchy and KP hierarchy*, J. Math. Phys. **35** (1994) 5844-5849.
- [18] F. Nijhoff, O. Ragnisco and V. Kuznetsov, *Integrable Time-Discretisation of the Ruijsenaars-Schneider Model*, Commun. Math. Phys. **176** (1996) 681-700.
- [19] A. Kuniba, T. Nakanishi and J. Suzuki, *T-systems and Y-systems in integrable systems*, J. Phys. A **44** (2011) 103001.
- [20] A.G. Izergin, *Partition function of the six-vertex model in a finite volume*, Soviet Phys. Dokl. **32** (1987) 878-879;  
 O. Foda, M. Wheeler and M. Zuparic, *XXZ scalar products and KP*, Nucl. Phys. **B820** (2009) 649;  
 K. Takasaki, *KP and Toda tau functions in Bethe ansatz*, Proceedings of the Infinite Analysis 09, New trends in quantum integrable systems, B. Feigin, M. Jimbo, M. Okado (eds), 2011, World Scientific, arXiv:1003.3071 [math-ph].